Homework set 5

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1 Problem 1 - Iterative solvers- the Jacobi method

2 Problem 2 - The unsteady diffusion equation

In this problem you will solve the following partial differential equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} + S(x,t), \tag{1}$$

over a domain of length $L = 1$, using $\kappa = 0.1$ and 40 gridpoints $dx = L/40$ and $S(x) = 0$

(a) Write a script **diffFE.m** to solve the diffusion equation above with a forward Euler time discretization. Use first a timestep $dt = 0.0006$ and then a timestep of $dt = 0.06$ for a total time of 1.2. Plot and compare the two solutions, why are they not identical ?

(b) Write a second script **diffImplicit.m** that solves the same problem with a fully implicit time discretization. At each time step, the algebrais system is solved with the thomas algorithm solver you implemented in Homework set 3. Use the same two timesteps as before and plot you results. Explain the difference between the forward Euler and the implicit method.

Turn in a copies of each script, the plots and discussion.

3 Problem 3 - The Stefan problem

The Stefan problem refers to a simple diffusion and phase change problem and can be solved analytically. Here we will assume that the domain $0 \le x \le L$ is initially a pure solid at its melting temperature $T_m = 0$. The temperature is instantaneously raised to $T_b = 1$ at its left boundary ($x = 0$) for all subsequent time. We will consider the other boundary (right) to be fixed at $T = T_m$ at all time. The equations that describe the physical system are

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - \frac{L}{c}\frac{\partial \phi}{\partial t}, \tag{2}$$

where $L = 1$ is the latent heat, c is the specific heat and $\phi(x,t)$ the melt fraction (between 0 for pure solid to 1 for pure liquid) at time t and position x. We introduce the Stefan number $St = c(T_b - T_m)/L$ which is a measure of the melting efficiency of the system. As seen in class, we will use the Enthalpy method which uses an iterative method to solve for this problem. At time t, the unsteady diffusion equation (see equation (1)) is solved first with

$$S(x) = -\frac{L}{c}\left(\phi^{k-1}(x,t) - \phi(x,t-dt)\right), \tag{3}$$

where $\phi^k(x,t)$ is the (k-1)th iteration for the solution of the melt fraction at timestep equivalent to time t. Equations (2) and (3) allow you to compute the approximated solution of T (T^k) at the kth iteration. We will use $T^k(x,t)$ to compute the Enthalpy

$$H(x,t) = cT^k(x,t) + L\phi^{k-1}(x,t). \tag{4}$$

This allows us to compute in turn the approximation of the melt fraction at the kth iteration at t

$$\phi^k(x,t) = \begin{cases} 0 & \text{if } H(x,t) \leq cT_m \\ \frac{H(x,t)-cT_m}{L} & \text{if } cT_m < H(x,t) < cT_m + L. \\ 1 & \text{if } H(x,t) \geq cT_m + L \end{cases} \tag{5}$$

To summarize at each timestep the following sequence of calculation is repeated until convergence for each timestep (after convergence, move to the next timestep):

1. using the melt fraction ϕ^{k-1} from the previous iteration (if this is the first iteration of the given timestep use the melt fraction that you converged to in the previous timestep) to compute the new $T^k(x,t)$ - use the Thomas algorithm with the source term given by equation (3).
2. using the new $T^k(x,t)$ compute the Enthalpy $H(x,t)$ still using $\phi^{k-1}(x,t)$.
3. Compute the updated melt fractions $\phi^k(x,t)$ from the Enthalpy (equation 5).
4. test convergence, here we will use $norm(\phi^k - \phi^{k-1}) < Tol$, where $Tol = 1e-10.$, if this condition is satisfied, we proceed to the next timestep, otherwise we stay at the same timestep and repeat new iterations until convergence is achieved.

Write a script **stefan.m** that solves the unsteady diffusion equation with phase change described by equation (2). During the iterative process, the Thomas algorithm is used to solve for the diffusion equation. The script should require a few inputs initially, the number of gridpoints (we fix the length of the domain $Lx = 1$) N_x, the Stefan number St (here the latent heat is fixed at $L = 1$, $T_m = 0$ and $T_b = 1$, so St is used to fix c) and finally it requires a duration for the calculation t_{final} and number of timesteps N_t. Use $N_x = 50$, $St = 3$, $t_{final} = 0.4$, $N_t = 100$ and $\kappa = 0.1$. Show your result ($T(x, t_{final})$) together with the analytical solution given by

$$T(x,t) = \begin{cases} 1 - \frac{\mathrm{erf}(x/(2\sqrt{\kappa t}))}{\mathrm{erf}(0.9138)} & \text{if } T(x,t) \geq T_m \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

where erf is the error function.